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Accuracy of Point Cyclic Reductions for Poisson's Equation*

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Point cyclic reduction is a very fast method for solving Helmholtz's equation on a rectangle. However, it involves forming an approximation of the discretized equations, and by a fairly simple analysis we show conclusively that for the special case of Poisson's equation on a square the solution computed by point cyclic reductions does not converge to the continuum solution as the gridsize goes to zero.

1. INTRODUCTION

Point cyclic reduction (PCR) is an interesting idea discovered by Detyna [2] and developed into a very fast N^2-algorithm for obtaining an approximate solution to the five-point discretization of Helmholtz's equation on a rectangle. It is the purpose of this paper to analyze the accuracy of the PCR method for the special case of Poisson's equation on a square by looking at the Fourier components of the numerical solution. In fact, what we discover is that the solution computed (in exact arithmetic) by PCR does not converge to the continuum solution as the gridsize tends to zero. In other words there is a limit to the attainable accuracy regardless of how fine the grid is. For example, for the problem with solution $\sin x \sin 3y$ the error cannot be reduced below 60%.

Although this paper is self-contained, familiarity with the paper of Detyna [2] would be advantageous.

2. DERIVATION OF POINT CYCLIC REDUCTION

We consider Poisson's equation

$$\nabla^2 \phi = \rho$$

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for $0 \leqslant x \leqslant \pi$ and $0 \leqslant y \leqslant \pi$ with ϕ specified on the boundary of this square. Let $N = 2^M$ and the gridsize $h = \pi/N$. For $n = 1, 2, 4, \ldots, N/2, N$ define grids

$$G^{n+} = \{(inh, jnh) \mid 0 \leqslant i, j \leqslant N/n\},$$
$$G^{n\times} = \{(inh, jnh) \mid 0 \leqslant i, j \leqslant N/n; i + j \text{ even}\}.$$

Thus the points of G^{n+} form squares of side nh and the points of $G^{n\times}$ form diamonds of side $\sqrt{2}nh$. As in Detyna [2] we define the operator S^{n+} for a function on G^{n+} to be the summation of the function values at the four neighbouring meshpoints, and the similarly for $S^{n\times}$. In other words, $S^{n+} - 4$ and $S^{n\times} - 4$ are, respectively, the five-point difference operator on G^{n+} and the rotated five-point difference operator on $G^{n\times}$, and consequently

$$(nh)^{-2}(S^{n+} - 4) = \nabla^2 + O((nh)^2)$$

and

$$(\sqrt{2}nh)^{-2}(S^{n\times} - 4) = \underset{\vee}{\nabla^2} + O((\sqrt{2}nh)^2).$$

The object is to solve

$$h^{-2}(S^{+} - 4)\,\phi^{+} = \rho$$

for ϕ^{+} on the interior of G^{+} with ϕ^{+} given on the boundary of G^{+}. The idea of PCR is to reduce the set of equations from finer to coarser grids using the following pair of identities discovered by Detyna:

$$\begin{aligned} S^{2n+} - 4 &= (S^{n+} + 4)(S^{n+} - 4) - 2(S^{n\times} - 4), \\ S^{2n\times} - 4 &= (S^{n\times} + 4)(S^{n\times} - 4) - 2(S^{2n+} - 4). \end{aligned} \tag{1}$$

Let

$$\begin{aligned} \rho^{n+} &\stackrel{\text{def}}{=} (nh)^{-2}(S^{n+} - 4)\,\phi^{+}, \\ \rho^{n\times} &\stackrel{\text{def}}{=} (\sqrt{2}nh)^{-2}(S^{n\times} - 4)\,\phi^{+}. \end{aligned} \tag{2}$$

If the values ρ^{+} and $\rho^{\times}$ are known, the pair of identities (1) can be applied to compute ρ^{n+}, $\rho^{n\times}$ recursively for $n = 2, 4, 8, \ldots, N/2$ according to

$$\rho^{2n+} = (1 + \tfrac{1}{4}S^{n+})\,\rho^{n+} - \rho^{n\times} \tag{3a}$$

on the interior of G^{2n+} and

$$\rho^{2n\times} = (1 + \tfrac{1}{4}S^{n\times})\,\rho^{n\times} - \rho^{2n+} \tag{3b}$$

on the interior of $G^{2n\times}$. Having obtained $\rho^{N/2+}$, we can solve for ϕ^{+} on the interior point of $G^{N/2+}$ from

$$\phi^{+} = \frac{1}{4}\left(S^{N/2+}\phi^{+} - \left(\frac{N}{2}h\right)^{2}\rho^{N/2+}\right), \tag{4}$$

since $S^{N/2+}\phi^+$ involves only values of ϕ^+ on the boundary, which are known. (Or we could obtain ϕ^+ from $\rho^{N/2\times}$ or use some linear combination of these two equations.) Then we can solve for ϕ^+ on those interior points of $G^{N/2+}$ from

$$\phi^+ = \frac{1}{4}\left(S^{N/4\times}\phi^+ - \left(\sqrt{2}\frac{N}{4}h\right)^2 \rho^{N/4\times}\right),$$

since $S^{N/4\times}\phi^+$ involves only values of ϕ^+ on $G^{N/2+}$, which are known. And this can be continued until ϕ^+ is determined on all of G^+.

The values of ρ^+ are known to be ρ because of the way in which ϕ^+ is defined, but the values of $\rho^\times$ are not known. Detyna [2] suggests using the approximation

$$\tfrac{1}{8}(S^+ + 4)\rho^+ = \rho^\times + O(h^2).$$

It is not entirely obvious why this approximation is used instead of ρ^+, say, because both give an $O(h^2)$ approximation. An argument for this, based on Fourier analysis, is given in Section 4.

3. Stability

The $O(h^2)$ approximation error for the discrete Laplacian of ϕ^+ on $G^\times$ does not seriously affect the computation if it is numerically stable. Hence we do a simple stability analysis for the somewhat special situation where an error δ is introduced into the computation of $\rho^\times$:

$$\tilde{\rho}^+ \overset{\text{def}}{=} \rho^+,$$

$$\tilde{\rho}^\times \overset{\text{def}}{=} \rho^\times + \delta,$$

and for $n = 1, 2, 4, \ldots, N/4$, $\tilde{\rho}^{2n+}, \tilde{\rho}^{2n\times}$ are computed in exact arithmetic from $\tilde{\rho}^{n+}, \tilde{\rho}^{n\times}$ using recursion (3). We then look at the error $\varepsilon^{N/2+}$, where $\phi^+ + \varepsilon^{N/2+}$ is the value obtained from $\tilde{\rho}^{N/2+}$ and the boundary values of ϕ^+ according to (4). This analysis covers both the approximation error in $\frac{1}{8}(S^+ + 4)\rho^+ \simeq \rho^\times$ and the rounding error in this computation. We expect that the results of this analysis will also suggest to us the effects of subsequent roundoff errors.

For $n = 1, 2, 4, \ldots, N/2$ define ε^{n+} to be the error in a solution obtained from $\tilde{\rho}^{n+}$ by solving the discrete Poisson equation on G^{n+}; that is,

$$(nh)^{-2}(S^{n+} - 4)(\phi^+ + \varepsilon^{n+}) = \tilde{\rho}^{n+}$$

on the interior of G^{n+} with $\varepsilon^{n+} = 0$ on the boundary of G^{n+}. Similarly define $\varepsilon^{n\times}$. By definition of ρ^{n+} we then have

$$(nh)^{-2}(S^{n+} - 4)\,\varepsilon^{n+} = \tilde{\rho}^{n+} - \rho^{n+}. \tag{5}$$

Clearly Eqs. (3a) and (3b) are satisfied by $\tilde{\rho}^{n+} - \rho^{n+}$ and $\tilde{\rho}^{n\times} - \rho^{n\times}$, respectively; and so substituting (5) into these equations we get an equation in terms of ε^{2n+}, ε^{n+}, and $\varepsilon^{n\times}$ and an equation in terms of $\varepsilon^{2n\times}$, $\varepsilon^{n\times}$, and ε^{2n+}, which, using identify (1), simplify to

$$\begin{aligned}(S^{2n+}-4)(\varepsilon^{2n+}-\varepsilon^{n\times}) &= -(S^{n+}+4)(S^{n+}-4)(\varepsilon^{n\times}-\varepsilon^{n+}),\\(S^{2n\times}-4)(\varepsilon^{2n\times}-\varepsilon^{2n+}) &= -(S^{n\times}+4)(S^{n\times}-4))(\varepsilon^{2n+}-\varepsilon^{n\times}).\end{aligned}\tag{6}$$

Finally we have

$$\begin{aligned}\varepsilon^{+} &= 0,\\(\sqrt{2}h)^{-2}(S^{\times}-4)\,\varepsilon^{\times} &= \delta.\end{aligned}\tag{7}$$

Equation (6) is the key to a relatively easy stability analysis.

Let us look at a typical Fourier component of δ, and so let

$$\delta(x, y) = \bar{\delta}\sin px \sin qy$$

for x, y on the mesh G^{+} where $1 \leqslant p, q \leqslant N-1$. We restrict both p and q to be odd, for otherwise, δ vanishes at the interior point $(\pi/2, \pi/2)$ of $G^{N/2+}$. For scalar multiples of $\sin px \sin qy$ we have

$$S^{n+}\cdots = (2\cos pnh + 2\cos qnh)\cdots$$

and

$$S^{nx}\cdots = (4\cos pnh \cos qnh)\cdots.$$

Substituting into (6) and (7), we get

$$\varepsilon^{+} = 0,$$

$$\varepsilon^{\times} = -\frac{h^2}{\sin^2 ph + \sin^2 qh + (\cos ph - \cos qh)^2}\,\delta,$$

$$\varepsilon^{2n+}-\varepsilon^{n\times} = -\left(2+\frac{(\cos pnh-\cos qnh)^2}{\sin^2 pnh+\sin^2 qnh}\right)(\varepsilon^{n\times}-\varepsilon^{n+}),\tag{8a}$$

$$\varepsilon^{2n\times}-\varepsilon^{2n+} = -\left(2+\frac{2}{\cot^2 pnh+\cot^2 qnh}\right)(\varepsilon^{2n+}-\varepsilon^{n\times}).\tag{8b}$$

Clearly the multiplicative factors in (8a) and (8b) are each no less than 2 in absolute value. It is not difficult to show then that for $\varepsilon^{\times} \neq 0$

$$\frac{\varepsilon^{n\times}-\varepsilon^{n+}}{\varepsilon^{\times}} \geqslant n^2,$$

$$\frac{\varepsilon^{2n+}-\varepsilon^{n+}}{\varepsilon^{n\times}-\varepsilon^{n+}} \leqslant -1,$$

$$\frac{\varepsilon^{2n+}}{\varepsilon^{\times}} \leqslant -n^2-\left(\frac{n}{2}\right)^2-\cdots-1,$$

and hence for any $\varepsilon^{\times}$

$$|\varepsilon^{N/2+}| \geqslant \frac{\pi^2}{12} h^{-2} |\varepsilon^{\times}|. \tag{9}$$

With $p = 1$ and $q = 1$ we have that $|\varepsilon^{\times}| \geqslant \frac{1}{2}|\delta|$ and in this case the error δ is amplified by a factor like h^{-2}.

Detyna [2] gives an informal analytic argument suggesting the amplification factor is no worse than h^{-2} and gives empirical evidence suggesting a factor of $h^{-1.7}$ ($1.7 \cong \log_2(13.17/4)$). Here we have shown conclusively that the factor is at least h^{-2}, and in the Appendix we show rigorously that it is no worse than this.

4. Convergence

We can use the δ in the stability analysis to represent the error in the approximation we are using for $\rho^{\times}$:

$$\delta = [1 + \alpha(S^{+} - 4)]\,\rho^{+} - \rho^{\times}, \tag{10}$$

where $\alpha = \frac{1}{8}$. In order to see why $\alpha = \frac{1}{8}$ is the best choice, we will for the moment allow α to be any real number. Combining Eqs. (2), (7), and (10) we have

$$(S^{\times} - 4)\,\varepsilon^{\times} = 2\{[1 + \alpha(S^{+} - 4)](S^{+} - 4) - (S^{\times} - 4)\}\,\phi^{+}.$$

Assuming now that $\phi(x, y) = \sin px \sin qy$, we have from $h^{-2}(S^{+} - 4)\,\phi^{+} = \nabla^2\phi$ that

$$\phi^{+} = \frac{(ph/2)^2 + (qh/2)^2}{\sin^2 ph/2 + \sin^2 qh/2}\,\phi$$

and

$$\varepsilon^{\times} = -\tfrac{1}{2}\,\frac{(\cos ph - \cos qh)^2 + (8\alpha - 1)(2 - \cos ph - \cos qh)^2}{(\cos ph - \cos qh)^2 + \sin^2 ph + \sin^2 qh}\,\phi^{+}.$$

We note that for $p = q = N - 1$ this second factor is $4(1 - 8\alpha)\,h^{-2} + O(1)$, which is very large unless $\alpha = 1/8$. With this choice for α the approximation error in (10) vanishes for $p = q$ and is less than 50% in the worst case when $|p - q| = N - 2$.

With $\alpha = 1/8$ we look at the convergence of this discretization. For p and q fixed as $h \to 0$ we have

$$|\varepsilon^{\times}| = \frac{h^2}{8}\,\frac{(p^2 - q^2)^2}{p^2 + q^2} + O(h^4)$$

and so from (9)

$$|\varepsilon^{N/2+}| \geqslant \frac{\pi^2}{96}\frac{(p^2-q^2)^2}{p^2+q^2} + O(h^2).$$

For example, with $p=1$ and $q=3$ we have $|\varepsilon^{N/2+}| \geqslant \pi^2/15 + O(h^2)$ and so the PCR solution at the centre point is in error by at least 60% for small (enough) h.

The choice $\phi(x, y) = \frac{1}{2}x(x-1)\, y(y-1) + \frac{1}{2}x^2(x^2-1)\, y^2(y^2-1)$, $0 \leqslant x,\ y \leqslant 1$, tested numerically by Detyna, yields a maximum relative error of less than 0.1% for $N=128$. This is quite remarkable, especially since the approximation error

$$\tfrac{1}{8}(S^+ + 4)\,\rho^+ - \rho^\times = \tfrac{1}{8}h^2(\phi^+_{xxxx} - 2\phi^+_{xxyy} + \phi^+_{yyyy}) + O(h^4)$$

does not seem particularly small in this case.

The analysis can be extended to show that the approximation based on obtaining ϕ^+ at the centre point from $(\sqrt{2}Nh/2)^{-2}](S^{N/2\times} - 4)\,\phi^+ = \rho^{N/2\times}$ is not convergent. Nonetheless it is still conceivable that convergence is possible by taking some linear combination (not depending on p and q) of the equations involving $\rho^{N/2+}$ and $\rho^{N/2\times}$.

Finally, we note that it does not seem possible to get convergence by defining ρ^+ and $\rho^\times$ in some other (computationally efficient) way in terms of ρ.

5. Conclusion

We have shown that point cyclic reduction is unsuitable for Poisson's equation. It may be possible to salvage the PCR idea but this remains to be demonstrated. The only known N^2-algorithms that are numerically stable are based on an iterative method, in particular, the multigrid method [1].

Also, the practical value of doing a fairly detailed error analysis has been illustrated.

Appendix

Here we show how the analysis of Detyna [2] can be enhanced in order to establish conclusively that the error growth factor is no worse than $O(h^{-2})$. Let

$$\delta^{n+} \overset{\text{def}}{=} \tilde{\rho}^{n+} - \rho^{n+}$$

and

$$\delta^{n\times} \overset{\text{def}}{=} \tilde{\rho}^{n\times} - \rho^{n\times}.$$

Then

$$\begin{bmatrix} \delta^+ \\ \delta^\times \end{bmatrix} = \begin{bmatrix} 0 \\ \delta \end{bmatrix}$$

and

$$\begin{bmatrix} \delta^{2n+} \\ \delta^{2n\times} \end{bmatrix} = \begin{bmatrix} 1+\frac{1}{4}S^{n+} & -1 \\ -1-\frac{1}{4}S^{n+} & 2+\frac{1}{4}S^{n\times} \end{bmatrix} \begin{bmatrix} \delta^{n+} \\ \delta^{n\times} \end{bmatrix}.$$

With δ equal to a scalar multiple of $\sin px \sin qy$, $1 \leqslant p$, $q \leqslant N-1$, we have $S^{n+} = 2(\xi+\eta)$ and $S^{n\times} = 4\xi\eta$, where $\xi = \cos pnh$ and $\eta = \cos qnh$, and so the matrix above becomes

$$A^{(n)} = \begin{bmatrix} 1+\frac{1}{2}(\xi+\eta) & -1 \\ -1-\frac{1}{2}(\xi+\eta) & 2+\xi\eta \end{bmatrix}.$$

The spectral radius of this matrix is maximized for $\xi = \eta = 1$ only, and the matrices of right and left eigenvectors, respectively, for $\xi = \eta = 1$ are

$$T = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \quad \text{and} \quad T^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

For general ξ and η we have that

$$T^{-1}A^{(n)}T = \begin{bmatrix} \frac{1}{4}+\frac{1}{3}(\xi+\frac{1}{2})(\eta+\frac{1}{2}) & \frac{3}{8}-\frac{2}{3}(\xi-\frac{1}{4})(\eta-\frac{1}{4}) \\ -\frac{1}{3}(\xi-1)(\eta-1) & \frac{5}{2}+\frac{2}{3}(\xi+\frac{1}{2})(\eta+\frac{1}{2}) \end{bmatrix},$$

and the infinity norm of this matrix is

$$\| \cdots \|_\infty = \max_i \sum_j |(\cdots)_{ij}| = 3 + \xi\eta \leqslant 4.$$

Furthermore

$$\begin{bmatrix} \delta^{N/2+} \\ \delta^{N/2\times} \end{bmatrix} = T(T^{-1}A^{(N/4)}T)\cdots(T^{-1}A^{(1)}T)\,T^{-1}\begin{bmatrix} 0 \\ \delta \end{bmatrix},$$

and so

$$\begin{aligned} |\delta^{N/2+}| &\leqslant \|\,\text{1st row of } T\|_\infty \cdot 4^{M-1} \cdot \left\| T^{-1}\begin{bmatrix} 0 \\ \delta \end{bmatrix} \right\|_\infty \\ &\leqslant \frac{\pi^2}{6} h^{-2} |\delta|. \end{aligned}$$

Finally we get

$$|\varepsilon^{N/2+}| = \left| -\frac{1}{4}\left(\frac{N}{2}h\right)^2 \delta^{N/2+} \right| \leqslant \frac{\pi^4}{96} h^{-2} |\delta|.$$

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